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Daugavet points and Δ-points in Lipschitz-free spaces

by

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Abstract. We study Daugavet points and Δ-points in Lipschitz-free Banach spaces. We prove that if M is a compact metric space, then $\mu \in S_{\mathcal{F}(M)}$ is a Daugavet point if and only if there is no denting point of $B_{\mathcal{F}(M)}$ at distance strictly smaller than 2 from μ. Moreover, we prove that if x and y are connectable by rectifiable curves of length as close to $d(x, y)$ as we wish, then the molecule $m_{x,y}$ is a Δ-point. Some conditions on M which guarantee that the previous implication reverses are also obtained. As a consequence, we show that Lipschitz-free spaces are natural examples of Banach spaces where we can guarantee the existence of Δ-points which are not Daugavet points.

1. Introduction. Let X be a Banach space and $T : X \to X$ be a bounded operator. We say that T satisfies the *Daugavet equation* if

$$\|T + I\| = 1 + \|T\|, \tag{1.1}$$

where I denotes the identity operator. The Daugavet equation has been widely studied in the literature ([19, 21, 26] and references therein), even in a more general context, where T is not necessarily a linear operator [11, 20, 27].

One of the most famous properties related to the Daugavet equation, which justifies its name, is the Daugavet property. Recall that a Banach space X has the *Daugavet property* if every rank-one operator satisfies the Daugavet equation. This property comes from [12], where it is proved that $C([0, 1])$ enjoys this property. Since then, a lot of examples of Banach spaces enjoying the Daugavet property have been found, such as $\mathcal{C}(K)$ for a perfect compact Hausdorff topological space K, $L_1(\mu)$ and $L_\infty(\mu)$ for a non-atomic measure μ, or the space of Lipschitz functions $\mathrm{Lip}_0(M)$ over a metrically convex space M (see [18, 21, 29] and the references therein for details).

In [17] the following weaker property is considered: a Banach space X is said to be a *space with bad projections* if $\|I - P\| \geq 2$ for every rank-one

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projection $P : X \to X$. This property was rediscovered later under the names of *LD2P+* [1] and of *diametral local diameter two property* [7].

One of the reasons why the above properties have attracted attention is that they have strong connections with the geometry of the unit ball of a Banach space. To be more precise, let us recall the following two results.

THEOREM 1.1 ([21, Theorem 2.1]). *Let X be a Banach space. The following assertions are equivalent:*

(1) *X has the Daugavet property.*
(2) *For every $x \in S_X$, every $\varepsilon > 0$ and every slice S of B_X there exists $y \in S$ such that*
$$\|x - y\| > 2 - \varepsilon.$$

THEOREM 1.2 ([17, Theorem 1.4]). *Let X be a Banach space. The following assertions are equivalent:*

(1) *X is a space with bad projections.*
(2) *For every $x \in S_X$, every $\varepsilon > 0$ and every slice S of B_X containing x there exists $y \in S$ such that*
$$\|x - y\| > 2 - \varepsilon.$$

The previous characterizations motivated the authors of [2] to introduce local versions of the above geometric properties, which will be central to the present paper. Given a Banach space X and a point $x \in S_X$, it is said that the point x is

- a *Daugavet point* if, given any slice S of B_X and any $\varepsilon > 0$, there exists $y \in S$ with $\|x - y\| > 2 - \varepsilon$;
- a Δ*-point* if, given any slice S of B_X containing x and any $\varepsilon > 0$, there exists $y \in S$ with $\|x - y\| > 2 - \varepsilon$.

In the previous language, a Banach space X has the Daugavet property (respectively, is a space with bad projections) if every point in S_X is a Daugavet point (respectively, a Δ-point). Deeper connections between Daugavet and Δ-points with the Daugavet property are exhibited in [2]. Furthermore, examples of Daugavet and Δ-points in some classical Banach spaces are exhibited in [2, Section 3].

The main aim of this paper is to study Daugavet points and Δ-points in Lipschitz-free spaces (see formal definition in Section 2). In the last years, geometric properties around the Daugavet property have been deeply studied [8, 14, 22, 24, 25]. Of particular interest to us is [14, Theorem 3.5] where it is proved that a Lipschitz-free space $\mathcal{F}(M)$ has the Daugavet property if and only if the metric space M is a *length space* (i.e. for any distinct points $x, y \in M$ their distance $d(x, y)$ is the infimum of the lengths of rectifiable curves in M joining them). We will take advantage of a localization of this property in order to obtain sufficient conditions for a molecule $m_{x,y}$ to be a

Δ-point in Section 4 which, in a large class of examples, will turn out to be equivalences. Furthermore, in Section 3 we obtain a characterization of when any element $\mu \in S_{\mathcal{F}(M)}$ is a Daugavet point in terms of a separation condition from the set of denting points. As a consequence of the above results, we show that Lipschitz-free Banach spaces are a class where one can easily and naturally produce examples of Δ-points which are not Daugavet points, in contrast with previously known examples which required the study of absolute sums of Banach spaces [2, Corollary 5.5] or technical constructions of Banach spaces with 1-unconditional bases [3, Theorem 3.1].

Let us now describe in more detail the content of the paper. In Section 2 we present necessary notation together with some preliminary results. In Section 3 we study Daugavet points in $\mathcal{F}(M)$ and in $\mathrm{Lip}_0(M)$. We prove in Theorem 3.2 that for a compact metric space M, an element $\mu \in S_{\mathcal{F}(M)}$ is a Daugavet point if μ is at distance 2 from every denting point of $B_{\mathcal{F}(M)}$. Furthermore, when μ is of the form $m_{x,y}$, we characterize the fact that μ is a Daugavet point in terms of a geometric condition on the points $x, y \in M$. We end the section with a brief discussion about Daugavet points in $\mathrm{Lip}_0(M)$ in connection with locality properties of Lipschitz functions. In Section 4 we define the concept of *connectable molecule* (Definition 4.1), which can be seen as a localization of length property of metric spaces. We prove that if $m_{x,y}$ is a connectable molecule then it is a Δ-point. This permits us, on the one hand, to get a procedure to construct Lipschitz-free spaces with molecules which are Δ-points but not Daugavet points. On the other hand, in search of conditions under which molecules which are Δ-points are connectable we prove that a molecule $m_{x,y}$ is a Δ-point if and only if for a given slice S containing $m_{x,y}$ and $\varepsilon > 0$ there exist $u, v \in M$ with $0 < d(u, v) < \varepsilon$ such that $m_{u,v} \in S$ (Theorem 4.7). We end the section with some conditions under which every molecule which is a Δ-point is connectable in subsets of strictly convex Banach spaces (Theorems 4.13 and 4.17).

2. Notation and preliminary results. We will consider only real Banach spaces. Given a Banach space X, we denote the closed ball (respectively, the sphere) of X centered at $x \in X$ with radius $r > 0$ by $B_X(x, r)$ (respectively, $S_X(x, r)$). If the center is 0 and the radius is 1, we simply write B_X and S_X. We will also denote by X^* the topological dual of X. By a *slice of* B_X we mean a set of the form

$$S(x^*, \alpha) := \{x \in B_X : x^*(x) > \sup x^*(B_X) - \alpha\},$$

where $x^* \in X^*$ and $\alpha > 0$. If X is a dual Banach space, this set will be called a w^**-slice* if x^* belongs to the predual of X. Note that finite intersections of slices of C (respectively of w^*-slices of B_X) form a basis for the inherited weak (respectively weak-star) topology of B_X.

Let X be a Banach space and let $x \in S_X$. Recall that x is a *preserved extreme point* if x is an extreme point of $B_{X^{**}}$. Also, x is a *denting point* if there exist slices of B_X of arbitrarily small diameter containing x. We will denote by $\mathrm{dent}(B_X)$ the set of denting points of B_X.

It is known that Δ-points in a Banach space X can be characterized in the following way (see [2, Lemma 2.1]). Let $x \in S_X$ be given. The following assertions are equivalent:

(1) x is a Δ-point;
(2) for every slice S of B_X with $x \in S$ and $\varepsilon > 0$, there exists $y \in S$ such that $\|x - y\| \geq 2 - \varepsilon$;
(3) for every $x^* \in X^*$ with $x^*(x) = 1$, the projection $P = x^* \otimes x$ satisfies $\|I - P\| \geq 2$.

This result allows us to obtain the following further characterization of Δ-points, which is probably well known to specialists, but whose proof we include for completeness.

Lemma 2.1. *Let X be a Banach space and $x \in S_X$ be a Δ-point. For every $\varepsilon > 0$ and every slice $S = S(f, \alpha)$ of B_X with $x \in S$ and $\frac{\alpha}{1-\alpha} < \varepsilon$, there exists a slice $S(g, \alpha_1)$ of B_X such that $S(g, \alpha_1) \subset S(f, \alpha)$ and $\|x - z\| \geq 2 - \varepsilon$ for all $z \in S(g, \alpha_1)$.*

Proof. The proof will follow the lines of that of [21, Lemma 2.1]. Choose $\eta > 0$ so small that $\eta < 1 - \frac{1-\alpha}{f(x)}$ and $\eta < \varepsilon - \frac{\alpha}{1-\alpha}$. If $P := (f(x)^{-1} f) \otimes x$, then $\|I - P\| \geq 2$ by [2, Lemma 2.1]. It follows that there exists $y^* \in S_{X^*}$ such that $\|y^* - P^* y^*\| > 2 - \eta$. Define $g = \frac{P^* y^* - y^*}{\|P^* y^* - y^*\|} \in X^*$ and $\alpha_1 = 1 - \frac{2-\eta}{\|P^* y^* - y^*\|}$. If $z \in S(g, \alpha_1)$, then

$$y^*(x) \frac{f(z)}{f(x)} - y^*(z) > 2 - \eta.$$

We may assume that $y^*(x) > 0$ (since $y^*(x)$ cannot be zero). Then we get $y^*(x) \frac{f(z)}{f(x)} > 1 - \eta$, so $f(z) > (1 - \eta) f(x) > 1 - \alpha$. Moreover, $\|f(x)^{-1} x - z\| > 2 - \eta$ since $f(x)^{-1} y^*(x) - y^*(z) > 2 - \eta$. This implies that

$$\|x - z\| > (2 - \eta) - \alpha > 2 - \varepsilon. \blacksquare$$

By using [17, Lemma 2.1], we can improve the previous result.

Lemma 2.2. *Let X be a Banach space. Then $x \in S_X$ is a Δ-point if and only if for every $\varepsilon > 0$ and every slice $S = S(f, \alpha)$ of B_X with $x \in S$, there exists a slice $S(g, \alpha_1)$ of B_X such that $S(g, \alpha_1) \subset S(f, \alpha)$ and $\|x - z\| \geq 2 - \varepsilon$ for all $z \in S(g, \alpha_1)$.*

Proof. We only need to prove the "only if" part. Let $\varepsilon > 0$ and a slice $S = S(f, \alpha)$ of B_X with $x \in S$ be given. Assume, up to a normalization argument, that $\|f\| = 1$. Pick any $\eta > 0$ with $\eta < \alpha$ and $\frac{\eta}{1-\eta} < \varepsilon$. By [17, Lemma 1.4], there exists $h \in S_{X^*}$ such that $x \in S(h,\eta) \subset S(f,\alpha)$. Applying the above lemma to the slice $S(h,\eta)$ and $\varepsilon > 0$, we may find a slice $S(g,\alpha_1)$ of B_X such that $S(g,\alpha_1) \subset S(h,\eta)$ and $\|x-z\| \geq 2-\varepsilon$ for all $z \in S(g,\alpha_1)$. As $S(g,\alpha_1)$ is contained in $S(f,\alpha)$ as well, we are done. ∎

REMARK 2.3. Similar estimates to the ones of the previous two lemmas allow one to prove the following result: Let X be a Banach space and $x \in S_X$ be a Daugavet point. Then, for every slice S of B_X and every $\varepsilon > 0$, there exists a slice T of B_X contained in S and such that

$$\|x-z\| > 2-\varepsilon \quad \text{for every } z \in T.$$

REMARK 2.4. Let us explain our interest in Lemma 2.2. Let X be a Banach space and let A be a subset of B_X such that $\overline{\mathrm{co}}(A) = B_X$. Pick a Δ-point $x \in S_X$. By definition, given a slice S containing x, there are $y \in S$ such that $\|x-y\| > 2-\varepsilon$. Lemma 2.2 allows us to guarantee that one such element y can be found in A. Indeed, Lemma 2.2 implies the existence of a slice T contained in S such that every $y \in T$ satisfies $\|x-y\| > 2-\varepsilon$. Now, since $\overline{\mathrm{co}}(A) = B_X$, A intersects every slice of B_X, in particular $T \cap A \neq \emptyset$. This property is of relevance for Section 4 (in particular, for Lemma 4.6).

Let us now introduce the necessary notation for Lipschitz-free spaces together with a preliminary result. Given a metric space M and a point $x \in M$, we will denote by $B(x,r)$ (respectively, $S(x,r)$) the closed ball (respectively, sphere) centered at x with radius r. Given $x,y \in M$ we define the *metric segment* by

$$[x,y] := \{z \in M : d(x,y) = d(x,z) + d(y,z)\}.$$

Let M be a metric space with a distinguished point $0 \in M$. The couple $(M,0)$ is commonly called a *pointed metric space*. By an abuse of language we will say only "let M be a pointed metric space" and similar sentences. The vector space of Lipschitz functions from M to $\mathbb{R}$ will be denoted by $\mathrm{Lip}(M)$. Given $f \in \mathrm{Lip}(M)$, we denote its Lipschitz constant by

$$\|f\|_L = \sup\left\{\frac{|f(x)-f(y)|}{d(x,y)} : x,y \in M,\ x \neq y\right\}.$$

This is a seminorm on $\mathrm{Lip}(M)$ which is clearly a Banach space norm on the space $\mathrm{Lip}_0(M) \subset \mathrm{Lip}(M)$ of Lipschitz functions on M vanishing at 0.

We denote by δ the canonical isometric embedding of M into $\mathrm{Lip}_0(M)^*$, given by $\langle f, \delta(x)\rangle = f(x)$ for $x \in M$ and $f \in \mathrm{Lip}_0(M)$. We denote by $\mathcal{F}(M)$ the norm-closed linear span of $\delta(M)$ in the dual space $\mathrm{Lip}_0(M)^*$, which is usually called the *Lipschitz-free space over* M; for background, see the survey [15] and the book [28] (where it is named the "Arens–Eells space"). It is well known that $\mathcal{F}(M)$ is an isometric predual of $\mathrm{Lip}_0(M)$ [15, p. 91]. We will write $\delta_x := \delta(x)$ for $x \in M$, and use the name *molecule* for elements of

$\mathcal{F}(M)$ of the form

$$m_{x,y} := \frac{\delta_x - \delta_y}{d(x,y)}$$

for distinct $x, y \in M$. With a slight abuse of notation, we shall write $f(m_{x,y})$ for $\frac{f(x)-f(y)}{d(x,y)}$.

It is convenient to recall an important tool to construct Lipschitz functions, McShane's classical extension theorem. It says that if $N \subseteq M$ and $f\colon N \to \mathbb{R}$ is a Lipschitz function, then there is an extension to a Lipschitz function $F\colon M \to \mathbb{R}$ with the same Lipschitz constant; see for example [28, Theorem 1.33].

Let us now consider the following definitions.

Definition 2.5. Let M be a metric space and $f \in \mathrm{Lip}_0(M)$.

(1) We say that f is *local* if for every $\varepsilon > 0$ there exist $u \neq v$ in M with $d(u,v) < \varepsilon$ and $f(m_{u,v}) > \|f\| - \varepsilon$.
(2) We say that a point $t \in M$ is an *ε-point* of f if for every neighborhood $U \subset M$ of t, there exist $u \neq v$ in U such that $f(m_{u,v}) > \|f\| - \varepsilon$.
(3) We say that f is *spreadingly local* if, for every $\varepsilon > 0$, there are infinitely many ε-points of f.

The above definitions come from [18], where it was proved that if M is a compact metric space then if every Lipschitz function is local then $\mathrm{Lip}_0(M)$ has the Daugavet property. Later, in [14] it was proved that it is actually a characterization even when M is complete.

Let us end with the following preliminary lemma, motivated by the ideas around the results of [25, Section 3], which tells us that one way of finding molecules far from a given element of $\mathcal{F}(M)$ is to look for close enough points. Namely, we have the following result.

Theorem 2.6. *Let M be a metric space and let u_n, v_n be two sequences in M such that $u_n \neq v_n$ for every n and $d(u_n, v_n) \to 0$. Then, for every $\mu \in S_{\mathcal{F}(M)}$,*

$$\|\mu + m_{u_n,v_n}\| \to 2.$$

Proof. Assume for contradiction that there exist $\mu \in S_{\mathcal{F}(M)}$ and $\varepsilon_0 > 0$ such that

$$\|\mu + m_{u_n,v_n}\| \leq 2 - \varepsilon_0 \quad \text{for every } n \in \mathbb{N}.$$

Since linear combinations of evaluation mappings are dense in $\mathcal{F}(M)$ we can assume that $\mu = \sum_{i=1}^{n} \lambda_i \delta_{x_i}$ where $\{x_1, \ldots, x_n\} \subseteq M \backslash \{0\}$. Pick $f \in S_{\mathrm{Lip}_0(M)}$ with $f(\mu) = 1$ and define

$$N := \{x_1, \ldots, x_n\} \cup \{0\}.$$

Define $\theta := \inf_{x\neq y\in N} d(x,y) > 0$. Up to taking a subsequence, we can assume that $d(u_n,v_n) \le \frac{\theta}{2n}$ for every $n\in\mathbb{N}$. Hence

$$\begin{aligned} d(x,y)+d(u_n,v_n) &\le d(x,u_n)+d(y,v_n)+2d(u_n,v_n)\\ &\le d(x,u_n)+d(y,v_n)+\frac{1}{n}(d(x,y)+d(u_n,v_n)), \end{aligned}$$

so

$$(1-1/n)(d(x,y)+d(u_n,v_n)) \le d(x,u_n)+d(y,v_n)$$

for all distinct $x,y\in N$ and every $n\in\mathbb{N}$.

We define, for every $n\in\mathbb{N}$, a Lipschitz function g_n on $N\cup\{u_n,v_n\}$ as $g_n(x)=f(x)$ for every $x\in N$,

$$\begin{aligned} g_n(u_n) &:= \inf_{x\in N}\left(g_n(x)+\frac{1}{1-1/n}d(x,u_n)\right),\\ g_n(v_n) &:= \sup_{x\in N\cup\{u_n\}}\left(g_n(x)-\frac{1}{1-1/n}d(x,v_n)\right). \end{aligned}$$

Notice that $\|g_n\| \le \frac{1}{1-1/n}$ for every $n\in\mathbb{N}$ (see, for example, [28, Proposition 1.32]). Fix $n\in\mathbb{N}$ so large that $1-1/n > 1-\varepsilon_0/2$. By McShane's theorem, we can extend g_n to the whole M without increasing its Lipschitz norm. Since g_n agrees with f on N, we get $g_n(\mu)=1$. We claim that $g_n(u_n)-g_n(v_n)\ge d(u_n,v_n)$ for every $n\in\mathbb{N}$ (equivalently $g_n(m_{u_n,v_n})\ge 1$). By definition, there are $z\in N$ and $z'\in N\cup\{u_n\}$ such that $g_n(u_n) = f(z)+(1-1/n)^{-1}d(z,u_n)$ and $g_n(v_n) = g_n(z')-(1-1/n)^{-1}d(z',v_n)$. If $z'=u_n$, then $g_n(u_n)-g_n(v_n)=(1-1/n)^{-1}d(u_n,v_n)\ge d(u_n,v_n)$. If $z'\in N$, we have

$$\begin{aligned} g_n(u_n)-g_n(v_n) &= f(z)-f(z')+\frac{1}{1-1/n}(d(z,u_n)+d(z',v_n))\\ &\ge f(z)-f(z')+d(z,z')+d(u_n,v_n)\ge d(u_n,v_n) \end{aligned}$$

since $f(z)-f(z')+d(z,z')\ge 0$. Now

$$2-\varepsilon_0 \ge \|\mu+m_{u_n,v_n}\| \ge \frac{g_n(\mu+m_{u_n,v_n})}{\|g_n\|} \ge 2\left(1-\frac{1}{n}\right),$$

a contradiction. ■

3. Daugavet points. In this section we will focus on studying Daugavet points in $\mathcal{F}(M)$ as well as in $\mathrm{Lip}_0(M)$. Let us start with the following easy observation, which says that Daugavet points have to be far from the set of denting points.

PROPOSITION 3.1. *Let X be a Banach space and $x_0\in S_X$ be a Daugavet point. Then, for every $y\in\mathrm{dent}(B_X)$, we have $d(x_0,y)=2$.*

Proof. Suppose that there exist $y \in \operatorname{dent}(B_X)$ and $\varepsilon > 0$ such that $d(x_0, y) \leq 2 - \varepsilon$. Choose a slice S containing y so that $\operatorname{diam}(S) < \varepsilon/2$. Note that

$$d(x_0, z) \leq d(x_0, y) + d(y, z) < 2 - \varepsilon + \varepsilon/2 = 2 - \varepsilon/2$$

for every $z \in S$. This implies that x_0 cannot be a Daugavet point. ∎

In general, the converse of the previous proposition is false. Indeed, $\operatorname{dent}(B_{\ell_\infty}) = \emptyset$, so $d(x, y) = 2$ for every $x \in B_{\ell_\infty}$ and every $y \in \operatorname{dent}(B_{\ell_\infty})$. However, ℓ_∞ fails the Daugavet property (see e.g. [29, p. 78]), so there are elements in B_{ℓ_∞} which are not Daugavet points.

In spite of the previous example, we will prove that the previous behavior does not occur for the class of Lipschitz-free spaces over compact metric spaces.

THEOREM 3.2. *Let M be a compact metric space and $\mu \in S_{\mathcal{F}(M)}$. The following assertions are equivalent:*

(1) *μ is a Daugavet point.*
(2) *If $\nu \in \operatorname{dent}(B_{\mathcal{F}(M)})$ then $d(\mu, \nu) = 2$.*

Moreover, if μ is of the form $m_{x,y}$ for certain $x \neq y \in M$, then the previous two are equivalent to:

(3) *If $u, v \in M$ satisfy $[u, v] = \{u, v\}$ then*

$$d(x, y) + d(u, v) \leq \min\{d(x, u) + d(y, v), d(x, v) + d(y, u)\}.$$

Proof. It is clear from Proposition 3.1 that (1)⇒(2).

(2)⇒(1). Pick a slice $S = S(f, \alpha)$, where $f \in S_{\operatorname{Lip}_0(M)}$. Then we have two possibilities for f:

- f is not local. Then by [10, Lemma 3.13], f attains its norm at a molecule $m_{u,v}$ which is a strongly exposed point (in particular, it is a denting point). By the assumptions $\|\mu - m_{u,v}\| = 2$ and, in view of the norm attaining condition, $m_{u,v} \in S$, and we are done in this case.
- f is local. In that case, by definition, we can find sequences $u_n \neq v_n$ with $d(u_n, v_n) \to 0$ and $m_{u_n,v_n} \in S$ for every $n \in \mathbb{N}$. By Theorem 2.6 we get $\|\mu + m_{u_n,v_n}\| \to 2$, and we are done.

(2)⇔(3). Assume now that $\mu = m_{x,y}$. Note that distinct elements $u, v \in M$ satisfy $[u, v] = \{u, v\}$ if and only if $m_{u,v}$ is an extreme point of $B_{\mathcal{F}(M)}$ [5, Theorem 3.2], which is in turn equivalent to being a preserved extreme point since M is compact [4, Theorem 4.2], which is equivalent to being a denting point by [13, Theorem 2.4]. Moreover, by [6, Theorem 2.4], $\|m_{x,y} \pm m_{u,v}\| = 2$ is equivalent to the inequality $d(x, y) + d(u, v) \leq \min\{d(x, u) + d(y, v), d(x, v) + d(y, u)\}$.

From all these facts, (2) and (3) are equivalent. ∎

Next, we exhibit an example of a metric space M such that $\mathcal{F}(M)$ does not have the Daugavet property but there exists a Daugavet point $m_{x,y}$ in it.

EXAMPLE 3.3. Let $M := \{-1\} \cup [0,1] \subseteq \mathbb{R}$ and let $y = 0, x = 1$. Then $m_{x,y}$ is a Daugavet point. Indeed, if u, v are such that $\{z \in M : d(z,u) + d(z,v) = d(u,v)\} = \{u,v\}$ then, up to relabeling, $u = 0$ and $v = -1$. Moreover,

$$d(x,y)+d(u,v)=2\leq \min\{d(x,u)+d(y,v), d(x,v)+d(y,u)\}=\min\{2,2\}=2.$$

By Theorem 3.2, $m_{x,y}$ is a Daugavet point. However, it is easy to see that $\mathcal{F}(M)$ does not have the Daugavet property because it is clearly not a length space [14, Theorem 3.5].

Now we turn to a brief discussion of Daugavet points and their w^*-version (see Definition 3.5) on $\mathrm{Lip}_0(M)$. First of all, a local argument in [18, Theorem 3.1] yields, following the proof word-by-word, the following result.

PROPOSITION 3.4. *Let M be a complete metric space and let $f \in S_{\mathrm{Lip}_0(M)}$. If f is spreadingly local, then f is a Daugavet point.*

It is natural to wonder whether the previous conclusion holds if f is merely a local Lipschitz function. We do not know the answer. Note that the main difficulty in studying Daugavet points in spaces of Lipschitz functions is that no good description of the weak topology in $\mathrm{Lip}_0(M)$ is known (which makes it difficult to determine whether or not a Lipschitz function belongs to a given (weak) slice). For this reason, we turn to a weak-star-version of the concept of Daugavet point in the following sense.

DEFINITION 3.5. Let X be a Banach space. An element $x^* \in S_{X^*}$ is said to be a *w^*-Daugavet point* if, given a w^*-slice S of B_{X^*} and any $\varepsilon > 0$, there exists $y^* \in S$ such that $\|x^* - y^*\| > 2 - \varepsilon$.

Apart from being a natural generalization of the concept of Daugavet point, the previous definition has deep connections with Banach spaces enjoying the Daugavet property thanks to the celebrated work [21]. Indeed, [21, Lemma 2.2] says that a Banach space X has the Daugavet property if and only if every element of S_{X^*} is a w^*-Daugavet point.

The following theorem confirms, for the case of w^*-Daugavet point, our initial intuition about local Lipschitz functions. The proof will use ideas from [8, Theorem 2.4].

THEOREM 3.6. *Let M be a complete metric space. If $f \in S_{\mathrm{Lip}_0(M)}$ is local, then f is a w^*-Daugavet point.*

Proof. Assume that f is local. Let $S = S(\mu, \alpha)$ be a w^*-slice of $B_{\mathrm{Lip}_0(M)}$ with $\mu \in \mathcal{F}(M) \setminus \{0\}$ and $\alpha > 0$. For fixed $\varepsilon > 0$, our aim is to find $g \in S$ such that $\|f - g\| > 2 - \varepsilon$. Without loss of generality assume $\varepsilon \leq \min\{\alpha, 1\}$

and $\mu = \sum_{i=1}^{N} \beta_i \delta_{x_i}$ with $\|\mu\| = 1$, where $\beta_1, \ldots, \beta_N \in \mathbb{R}$ and $x_1, \ldots, x_N \in M \setminus \{0\}$.

Let $h \in S$ be such that $h(\mu) \geq (1+\varepsilon)(1-\alpha)$. Pick $r \in (0, \varepsilon)$ such that all the balls $B(0,r), B(x_1,r), \ldots, B(x_N,r)$ are at distance at least r from each other. Let $u, v \in M$ be such that $u \neq v$, $d(u,v) < \varepsilon r/2$ and $f(m_{u,v}) > 1 - \varepsilon r/2$. Such u and v exist because f is local.

We will now extend the 1-Lipschitz function h from $\{0\} \cup \{x_1, \ldots, x_N\}$ to a $(1+\varepsilon)$-Lipschitz function $\tilde{h}$ on $\{0\} \cup \{x_1, \ldots, x_N\} \cup \{u, v\}$ so that

$$\tilde{h}(v) - \tilde{h}(u) = d(u,v).$$

In order to define $\tilde{h}$ at u and v, let $a, b \geq 0$ be such that

$$h(x) - d(x,u) \leq h(u) - a \leq h(v) + b \leq h(x) + d(x,v)$$

and

$$(h(v) + b) - (h(u) - a) = d(u,v)$$

where $x = x_i$ if $B(x_i, r) \cap \{u, v\} \neq \emptyset$ for some $i \in \{1, \ldots, N\}$, and $x = 0$ if no such i exists. The existence of such a, b satisfying also $a, b \leq 2d(u,v)$ is immediate from the following facts:

$$h(x) - d(x,u) \leq h(u), \quad |h(v) - h(u)| \leq d(u,v), \quad h(v) \leq h(x) + d(x,v),$$

and

$$d(u,v) \leq d(v,x) + d(u,x).$$

Now we set

$$\tilde{h}(u) = h(u) - a \quad \text{and} \quad \tilde{h}(v) = h(v) + b.$$

It is straightforward to verify that $\tilde{h}$ is $(1+\varepsilon)$-Lipschitz on $\{0\} \cup \{x_1, \ldots, x_N\} \cup \{u, v\}$.

Now extend $\tilde{h}$ further to a $(1+\varepsilon)$-Lipschitz function to all of M, still denoted by $\tilde{h}$. Set $g := (1+\varepsilon)^{-1}\tilde{h}$. Then $g \in B_{\mathrm{Lip}_0(M)}$ and $g(\mu) = (1+\varepsilon)^{-1} h(\mu) > 1 - \alpha$. Hence $g \in S$. On the other hand,

$$\|f - g\| \geq f(m_{u,v}) - g(m_{u,v}) \geq \left(1 - \frac{\varepsilon r}{2}\right) + \frac{1}{1+\varepsilon} > 2 - \varepsilon.$$

Consequently, f is a w^*-Daugavet point. ■

4. Δ-points. In this section we turn to the study of Δ-points in Lipschitz-free spaces. We are motivated by from the results of [14], where a metric characterization of when $\mathcal{F}(M)$ has the Daugavet property is given in terms of a metric property depending only on M, the property of M being a length space. Taking a look at the definition of length spaces, we consider the following local concept of length space.

Definition 4.1. Let M be a metric space and $x \neq y$ in M. We say that the points x and y are *connectable* if given $\varepsilon > 0$ there exists a 1-Lipschitz mapping $\alpha : [0, d(x, y) + \varepsilon] \to M$ with $\alpha(0) = y$ and $\alpha(d(x, y) + \varepsilon) = x$.

Notice that a metric space M is a length space if and only if any distinct points are connectable. This property is equivalent to the fact that for all $x, y \in M$ and $\delta > 0$ the set

$$\mathrm{Mid}(x, y, \delta) := B\left(x, \frac{1+\delta}{2} d(x, y)\right) \cap B\left(y, \frac{1+\delta}{2} d(x, y)\right)$$

is non-empty (see [14, Lemma 3.2]).

Our interest in this definition is the following result.

Proposition 4.2. *Let M be a metric space and let $x \neq y$ in M be connectable. Then $m_{x,y}$ is a Δ-point.*

Proof. Pick a slice $S = S(f, \delta)$ with $\|f\| = 1$ containing $m_{x,y}$. Let us find $u \neq v$ such that $m_{u,v} \in S$ and $\|m_{x,y} - m_{u,v}\| \approx 2$. Assume with no loss of generality that $d(x, y) = 1$. Find $0 < \beta < \delta$ such that $f(m_{x,y}) > 1 - \beta$, take $\eta > 0$ with $\frac{1-\beta}{1+\eta} > 1 - \delta$ and let $\alpha : [0, 1+\eta] \to M$ be a 1-Lipschitz curve such that $\alpha(0) = y$ and $\alpha(1 + \eta) = x$. Then $f \circ \alpha : [0, 1 + \eta] \to \mathbb{R}$ is a 1-Lipschitz map, so it is differentiable almost everywhere. Moreover,

$$1 - \beta < f(\alpha(1 + \eta)) - f(0) = \int_0^{1+\eta} (f \circ \alpha)' \leq (1 + \eta)\|(f \circ \alpha)'\|_\infty,$$

so there exists $t_0 \in [0, 1 + \eta]$ such that $(f \circ \alpha)'(t_0) > \frac{1-\beta}{1+\eta} > 1 - \delta$. Now pick $\varepsilon > 0$. By the definition of derivative and the previous condition we can find $t \in [0, 1+\eta]$ with $0 < |t - t_0| < \varepsilon$ and such that $\frac{f(\alpha(t))-f(\alpha(t_0))}{t-t_0} > 1 - \delta$. Now

$$\begin{aligned} 1 - \delta < \frac{f(\alpha(t)) - f(\alpha(t_0))}{t - t_0} &= \frac{f(\alpha(t)) - f(\alpha(t_0))}{d(\alpha(t), \alpha(t_0))} \frac{d(\alpha(t), \alpha(t_0))}{t - t_0} \\ &\leq f(m_{\alpha(t),\alpha(t_0)})\|\alpha\|_L \leq f(m_{\alpha(t),\alpha(t_0)}), \end{aligned}$$

which implies that $m_{\alpha(t),\alpha(t_0)} \in S$. Moreover $d(\alpha(t), \alpha(t_0)) < \varepsilon$. Summarizing we have proved that we can find molecules in S where the defining points are arbitrarily close. By Theorem 2.6, $m_{x,y}$ is a Δ-point. ■

Remark 4.3. As a matter of fact, in Proposition 4.2, we demonstrated a stronger property of $m_{x,y}$ than that of being a Δ-point. Namely, for every slice S containing $m_{x,y}$ and for each $\mu \in S_{\mathcal{F}(M)}$, there is a $\nu \in S$ with $\|\mu - \nu\|$ is arbitrarily close to 2. Let us formalize this property which will come up again: Given a Banach space X, a point $u \in S_X$ is said to be *large slice connected* if for every slice S containing u, for every $\varepsilon > 0$ and every $v \in S_X$ there is a $w \in S$ with $\|v - w\| > 2 - \varepsilon$.

Next we aim to get conditions under which connectability is equivalent to the fact that the corresponding molecule is a Δ-point. However, let us first deduce from the previous proposition and from Theorem 3.2 that there are Δ-points that are not Daugavet points in the context of Lipschitz-free spaces. This establishes a major difference between the class of Lipschitz-free spaces and L_1-spaces, where every Δ-point is a Daugavet point [2, Theorem 3.1].

EXAMPLE 4.4. Let $0 < r < 1$, $M := [0,1] \times \{0\} \cup \{(0,r),(1,r)\} \subseteq (\mathbb{R}^2, \|\cdot\|_2)$ and consider $x := (1,0)$ and $y := (0,0)$. Notice that $m_{x,y}$ is a Δ-point because there exists an isometry $\alpha : [0,1] \to M$ connecting x and y (namely $\alpha(t) := (t,0)$ for every $t \in [0,1]$). However, it is not a Daugavet point. To see this, pick $u := (1,r)$ and $v := (0,r)$, and notice that $A := \{z \in M : d(u,z) + d(v,z) = d(u,v)\} = \{u,v\}$. In fact, given $z \in M \setminus \{u,v\}$ we have $z = (t,0)$ for some $t \in [0,1]$. Hence

$$d(u,z) + d(v,z) = \sqrt{t^2 + r^2} + \sqrt{(1-t)^2 + r^2} > t + (1-t) = 1 = d(u,v),$$

which proves that $A = \{u,v\}$. Moreover,

$$d(x,u) + d(y,v) = 2r < 2 = d(x,y) + d(u,v),$$

so by Theorem 3.2(3) we conclude that $m_{x,y}$ is not a Daugavet point.

REMARK 4.5. The results of [2, Section 3] show that, in many classical Banach spaces, the concepts of Δ- and Daugavet point coincide. The first example of a Δ-point which is not a Daugavet point [2, Example 4.7] required a study of absolute normalized norms (which was pushed quite for in [16]). See also [3] for more technical examples of Banach spaces containing Δ-points which are not Daugavet points.

Theorem 3.2 together with Proposition 4.2 provides easy procedures to obtain metric spaces whose Lipschitz-free space has Δ-points which are not Daugavet points.

Our aim is now to get necessary conditions for a molecule $m_{x,y}$ to be a Δ-point. We begin with the following preliminary lemma.

LEMMA 4.6. *Let $x \neq y$ in M and $f \in S_{\mathrm{Lip}_0(M)}$. If $m_{x,y}$ is a Δ-point, then given a slice $S(f,\varepsilon/2)$ of $B_{\mathcal{F}(M)}$ with $m_{x,y} \in S(f,\varepsilon/2)$, there exist $u \neq v$ in M with $f(m_{u,v}) > 1-\varepsilon/2$ such that $d(u,v) < \frac{2\varepsilon}{(1-\varepsilon)^2} d(x,y)$.*

Proof. We follow [14, proof of Lemma 3.7]. Set $g := \frac{f+f_{x,y}}{2}$, where $f_{x,y}$ is defined by

$$f_{x,y}(t) := \frac{d(x,y)}{2} \frac{d(t,y) - d(t,x)}{d(t,y) + d(t,x)}.$$

Since $f_{x,y}(m_{x,y}) = 1$, g satisfies $\|g\| \geq g(m_{x,y}) > 1 - \varepsilon/4$. Since $m_{x,y}$ is a Δ-point by Lemma 2.2, there exists a slice $S(h,\eta)$ of $B_{\mathcal{F}(M)}$ such that

$S(h,\eta) \subset \{\mu \in B_{\mathcal{F}(M)} : g(\mu) > 1-\varepsilon/4\}$ and $\|m_{x,y} - z\| \geq 2-\varepsilon/2$ for every $z \in S(h,\eta)$. Pick $u \neq v$ in M such that $m_{u,v} \in S(h,\eta)$. Then, in particular,

$$\|m_{x,y} - m_{u,v}\| > 2-\varepsilon.$$

On the one hand, note that $f(m_{u,v}) > 1-\varepsilon/2$ and $f_{x,y}(m_{u,v}) > 1-\varepsilon/2$. Since $f_{x,y}(m_{u,v}) > 1-\varepsilon$, from [14, Lemma 3.6] we get

$$(1-\varepsilon)\max\{d(x,u)+d(y,u), d(x,v)+d(y,v)\} < d(x,y).$$

To obtain the desired conclusion, we will prove that

$$(1-\varepsilon)(d(x,y)+d(u,v)) \leq \min\{d(x,u)+d(y,v), d(x,v)+d(y,u)\}.$$

Notice that $\|m_{x,y} + m_{v,u}\| > 2-\varepsilon$ implies that there exists $\varphi \in S_{\mathrm{Lip}_0(M)}$ such that $\varphi(x)-\varphi(y) > (1-\varepsilon)d(x,y)$ and $\varphi(v)-\varphi(u) > (1-\varepsilon)d(u,v)$. Hence

$$\begin{aligned} 1 \geq \frac{\varphi(x)-\varphi(u)}{d(x,u)} &= \frac{\varphi(x)-\varphi(y)+\varphi(v)-\varphi(u)+\varphi(y)-\varphi(v)}{d(x,u)} \\ &\geq \frac{(1-\varepsilon)(d(x,y)+d(u,v))-d(y,v)}{d(x,u)}, \end{aligned}$$

which yields $(1-\varepsilon)(d(x,y)+d(u,v)) \leq d(x,u)+d(y,v)$. Using the same argument taking into account that $f(x)-f(y) > (1-\varepsilon)d(x,y)$ and $f(u)-f(v) > (1-\varepsilon)d(u,v)$ we get $(1-\varepsilon)(d(x,y)+d(u,v)) \leq d(x,v)+d(y,u)$, as desired.

With the previous inequalities in mind, the conclusion of the lemma follows from the estimates in [14, Lemma 3.7]. ∎

From the previous lemma and [17, Lemma 1.4], we obtain the following characterization of the molecules which are Δ-points in a Lipschitz-free space.

THEOREM 4.7. *Let $x \neq y$ in M. Then $m_{x,y}$ is a Δ-point if and only if for every slice $S = S(f,\alpha)$ containing $m_{x,y}$ with $\alpha < 1$ and for every $\varepsilon > 0$, there exist $u, v \in M$ with $0 < d(u,v) < \varepsilon$ such that $m_{u,v} \in S$.*

Proof. Suppose that $m_{x,y}$ is a Δ-point. Pick $\varepsilon > 0$ and $0 < \beta < \alpha$ such that $\frac{4\beta}{(1-2\beta)^2}d(x,y) < \varepsilon$. By [17, Lemma 1.4] there exists $g \in S_{\mathrm{Lip}_0(M)}$ such that

$$m_{x,y} \in S(g,\beta) \subseteq S.$$

Since $g(m_{x,y}) > 1-\beta$, by Lemma 4.6 there are $u \neq v$ in M such that $g(m_{u,v}) > 1-\beta$ (and so $m_{u,v} \in S$) and

$$d(u,v) < \frac{4\beta}{(1-2\beta)^2}d(x,y) < \varepsilon,$$

as desired.

Conversely, let $S = S(f,\alpha)$ be a slice containing $m_{x,y}$ with $\alpha < 1$ and let $\varepsilon > 0$. Using the existence of such $u, v \in M$ repeatedly, we may find a sequence (u_n, v_n) of points with $m_{u_n,v_n} \in S$ such that $0 < d(u_n, v_n) \to 0$. By Theorem 2.6, we conclude that $m_{x,y}$ is a Δ-point. ∎

REMARK 4.8. In fact, what we observed in the second part of the previous proof is that $m_{x,y}$ is large slice connected (see Remark 4.3). In particular, for distinct x and y in M, the point $m_{x,y}$ is a Δ-point if and only if $m_{x,y}$ is large slice connected.

Even though the above is a complete characterization of when a given molecule $m_{x,y}$ in $\mathcal{F}(M)$ is a Δ-point, we would like to obtain a condition which only depends on the metric space M. In order to do so, we prove the following consequence of Proposition 4.7.

COROLLARY 4.9. *Let M be a complete metric space and let $m_{x,y}$ a Δ-point. Pick $0 < r < d(x,y)$. Then, for every $\varepsilon > 0$,*

$$B(x, r + \varepsilon) \cap B(y, d(x,y) - r + \varepsilon) \neq \emptyset.$$

In particular, if M is compact, then $S(x,r) \cap S(y, d(x,y) - r) \neq \emptyset$.

Proof. The proof follows the lines of [14, Lemma 3.4, (iii)⇒(i)]. Assume with no loss of generality that $d(x,y) = 1$. Assume that there exist $0 < r < 1$ and $\varepsilon_0 > 0$ such that

$$B(x, r + \varepsilon_0) \cap B(y, 1 - r + \varepsilon_0) = \emptyset,$$

and let us prove that $m_{x,y}$ is not a Δ-point. Notice that we can assume that $d(B(x, r+\varepsilon_0), B(y, 1-r+\varepsilon_0)) \geq \delta_0 > 0$. Now define $g_i, f_i : M \to \mathbb{R}$, $i = 1, 2$, by

$$g_1(t) := \max\left\{r - \frac{1}{1+\varepsilon_0} d(x,t), 0\right\}, \qquad f_1(t) = g_1(t) - f_1(0)$$

$$g_2(t) := \min\left\{-(1-r) + \frac{1}{1+\varepsilon_0} d(y,t), 0\right\}, \quad f_2(t) = g_2(t) - g_2(0).$$

Notice that $\|f_i\|_L \leq \frac{1}{1+\varepsilon_0}$ since Lipschitz norm does not increase under taking maxima and minima of Lipschitz functions [28, Proposition 1.32]. Define $f := f_1 + f_2$, which is a Lipschitz function. Also $f(x) - f(y) = 1 = d(x,y)$, so $\|f\| \geq 1$. It is clear from the construction that $\{z \in M : f_1(z) \neq 0\} \subseteq B(x, r + \varepsilon_0)$ and $\{z \in M : f_2(z) \neq 0\} \subseteq B(y, 1 - r + \varepsilon_0)$. Define the slice

$$S := \left\{\mu \in B_{\mathcal{F}(M)} : f(\mu) > \frac{1}{1+\varepsilon_0}\right\}.$$

Notice that if $m_{u,v} \in S$ then, up to relabeling the points u and v, it follows that $u \in B(x, r + \varepsilon_0)$ and $v \in B(y, 1 - r + \varepsilon_0)$ (because otherwise $f(m_{u,v}) \leq \max\{\|f_1\|_L, \|f_2\|_L\} \leq \frac{1}{1+\varepsilon_0}$). This implies that $d(u,v) \geq \delta_0$. By Proposition 4.7 we infer that $m_{x,y}$ is not a Δ-point, as desired. ∎

REMARK 4.10. Let M be a complete metric space. By combining Corollary 4.9 with [14, Lemma 3.2], we find that if $m_{x,y}$ is a Δ-point for any $x \neq y$

in M, then M is a length space. Now, we have the following equivalent statements:

(1) M is a length space;
(2) $m_{x,y}$ is a Δ-point for any $x \neq y$ in M;
(3) $\mathcal{F}(M)$ has the Daugavet property.

For the proof of (1)$\Leftrightarrow$(3), see [14, Theorem 3.5].

Remark 4.11. According to [2, Definition 5.1], a Banach space X is said to have the *convex-DLD2P* if $B_X = \overline{\mathrm{co}}(\Delta)$, where Δ is the set of Δ-points of B_X.

In general, the convex-DLD2P does not imply that every element of the unit sphere is a Δ-point [2, Corollary 5.6]. However, Remark 4.10 shows that if Δ contains the set $\{m_{x,y} : x \neq y \text{ in } M\}$ (in particular, $\mathcal{F}(M)$ would trivially have the convex-DLD2P) then $\mathcal{F}(M)$ even enjoys the Daugavet property.

As pointed out above, a (complete) metric space M is a length space if and only if for all $x, y \in M$ and $\varepsilon > 0$,

$$B\left(x, \frac{d(x,y)}{2} + \varepsilon\right) \cap B\left(y, \frac{d(x,y)}{2} + \varepsilon\right) \neq \emptyset.$$

This might suggest that a local version could be true; in other words, that the converse of Corollary 4.9 holds. However, the following example, due to Luis García-Lirola, shows that this is not the case.

Example 4.12. Let $M := \{0, 1, x_t : t \in [0,1]\} \subset (\mathbb{R}, d)$ with the metric $d(x_t, x_s) = \min\{t + s, 2 - t - s\}$ for $t \neq s$, where $x_0 = 0$ and $x_1 = 1$. Then M is complete. It is clear that $B(0, r) \cap B(1, 1 - r) = \{x_r\} \neq \emptyset$ for every $0 < r < 1$. However, $m_{0,1}$ is not a Δ-point. Indeed, assume that it is. For $\alpha \in (0, 1/2)$, consider the map f defined as $f(x_t) = 0$ for $0 \leq t < \alpha$ and $f(x_t) = 1 - \alpha$ for $1 - \alpha < t \leq 1$. Observe that the slope of f is 1. Now, extend f by McShane to a Lipschitz map $\tilde{f}$ on M. Notice that $m_{0,1} \in S(f, 2\alpha)$ and there exist sequences u_n, v_n of points in M with $m_{u_n,v_n} \in S(f, 2\alpha)$ such that $0 < d(u_n, v_n) \to 0$. By the definition of the metric space M, both u_n and v_n converge to 0 or 1. However, in either case, $f(m_{u_n,v_n}) \to 0$, which is a contradiction.

In order to obtain a kind of converse of Proposition 4.2, our strategy will be to work with complete metric spaces M included in Banach spaces (so any $x, y \in M$ can be joined by geodesics in X) and then assume that the $\mathrm{diam}(B(x, r + \varepsilon) \cap B(y, d(x,y) - r + \varepsilon))$ tends to 0 as $\varepsilon \to 0$, in order to guarantee that M contains curves which are close to the geodesic which exists in X. The first result along these lines requires compactness of M but a very natural condition on X.

THEOREM 4.13. *Let X be a Banach space and M a compact subset of X. Assume that $\frac{x-y}{\|x-y\|}$ is an extreme point of B_X. The following assertions are equivalent:*

(1) *$m_{x,y}$ is a Δ-point.*
(2) *For every $0 < r < \|x-y\|$,*

$$S(x,r) \cap S(y, \|x-y\| - r) \neq \emptyset.$$

(3) *$[x,y] \subseteq M$. In particular, x and y are connectable by an isometric curve.*

Proof. (3)⇒(1) follows from Proposition 4.2, and (1)⇒(2) from Corollary 4.9. So it remains to prove that (2) implies (3). To this end, pick $0 < r < \|x-y\|$ and define $A := S(x,r) \cap S(y, \|x-y\| - r)$. Since $\frac{x-y}{\|x-y\|}$ is an extreme point, up to a shift and a normalization argument, [9, Lemma 2.1] implies that $S_X(x,r) \cap S_X(y, \|x-y\| - r)$ only contains one point, which is precisely $\big(\frac{r}{\|x-y\|}\big)x + \big(1 - \frac{r}{\|x-y\|}\big)y$. Since $A \neq \emptyset$ and $A \subseteq S_X(x,r) \cap S_X(y, \|x-y\| - r)$, we find that $\big(\frac{r}{\|x-y\|}\big)x + \big(1 - \frac{r}{\|x-y\|}\big)y \in A$, so it is in M. ∎

Let us end the section by obtaining a version of Theorem 4.13 which, assuming a stronger condition on X, will allow us to remove the compactness assumption on M. Let us consider the following definition.

DEFINITION 4.14. A Banach space X is said to be *midpoint locally uniformly rotund* (for short, MLUR) if whenever (x_n) and (y_n) are sequences in S_X and $\frac{1}{2}(x_n + y_n)$ converges to some element in S_X, then $\|x_n - y_n\| \to 0$.

It is not difficult to check that X is MLUR if and only if whenever (x_n) and (y_n) are sequences in X such that $\|x_n\|$ and $\|y_n\|$ tend to 1 and $\frac{1}{2}(x_n+y_n)$ converges to some member of S_X, it follows that $\|x_n - y_n\| \to 0$. Recall that a locally uniformly convex Banach space is MLUR and a MLUR Banach space is strictly convex. For more about rotundity in Banach spaces, see [23].

The following lemma is easy to check.

LEMMA 4.15. *Let X be a Banach space. Then X is MLUR if and only if whenever (x_n) and (y_n) are sequences in X such that $\|x_n\|$ and $\|y_n\|$ tend to 1 and $rx_n + (1-r)y_n$ converges to some member of S_X for some $0 < r < 1$, it follows that $\|x_n - y_n\| \to 0$.*

Now we are ready to obtain the following result.

LEMMA 4.16. *Let X be a MLUR Banach space, and $x \neq y$ in X. Then for $0 < r < d(x,y)$,*

$$\operatorname{diam}\left(B_X\left(x, r + \frac{1}{n}\right) \cap B_X\left(y, \|x-y\| - r + \frac{1}{n}\right)\right) \to 0$$

as $n \to \infty$.

Proof. By scaling and translating, we may assume that $x \in S_X$ and $y = -x$. Given $0 < r < 2$, let $x_n \in B_X(x, r+1/n) \cap B_X(-x, 2-r+1/n)$ for each $n \in \mathbb{N}$. In other words, $\|x_n - x\| \leq r + 1/n$ and $\|x_n + x\| \leq 2 - r + 1/n$ for every $n \in \mathbb{N}$. Then

$$2 \leq \|x_n - x\| + \|x_n + x\| \leq 2 + 2/n \to 2;$$

hence, passing to a subsequence if necessary, $\|x_n - x\| \to r$ and $\|x_n + x\| \to 2-r$ as $n \to \infty$. Let $z_n := (r + \frac{1}{n})^{-1}(x - x_n)$ and $w_n = (2 - r + \frac{1}{n})^{-1}(x + x_n)$ in B_X. Then $\|z_n\| \to 1$, $\|w_n\| \to 1$ and $\frac{r}{2} z_n + \left(1 - \frac{r}{2}\right) w_n \to x \in S_X$. By Lemma 4.15, we conclude that $\|z_n - w_n\| \to 0$ as $n \to \infty$. This implies that $x_n \to (1-r)x$ as $n \to \infty$. ∎

Now we are ready to get the desired result.

THEOREM 4.17. *Let X be a MLUR Banach space and M a complete subset of X. For $x \neq y$ in M, the following assertions are equivalent:*

(1) *$m_{x,y}$ is a Δ-point.*
(2) *For every $0 < r < \|x - y\|$,*

$$B(x, r + \varepsilon) \cap B(y, \|x - y\| - r + \varepsilon) \neq \emptyset.$$

(3) *$[x, y] \subseteq M$. In particular, x and y are connectable by an isometric curve.*

Proof. (3)⇒(1) follows from Proposition 4.2, and (1)⇒(2) from Proposition 4.9. So it remains to prove that (2) implies (3). To this end, let $x_n \in B(x, r + 1/n) \cap B(y, \|x - y\| - r + 1/n)$ for each $n \in \mathbb{N}$. Note from Lemma 4.16 that

$$\operatorname{diam}\left(B\left(x, r + \frac{1}{n}\right) \cap B\left(y, \|x - y\| - r + \frac{1}{n}\right)\right) \to 0$$

as $n \to \infty$. This implies that x_n converges to some x_0 in M (since M is complete). Moreover, $\|x_0 - x\| = r$ and $\|x_0 - y\| = \|x - y\| - r$. The strict convexity of X forces $x_0 = \left(1 - \frac{r}{\|x-y\|}\right)x + \left(\frac{r}{\|x-y\|}\right)y$. This proves that the segment $[x, y]$ is contained in M. ∎

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